MODELS OF CONTINUOUS MEDIA WITH INTERNAL DEGREES OF FREEDOM*

PMM Vol.32, №5, 1968, pp.771-785
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(Received May 6, 1968)

As we know, there is a need in modern physics and mechanics for the construction, analysis and utilization of new models of bodies with complicated properties.

Researchers are now in a position to begin the actual development of the macroscopic theory which requires investigation not only of gas motion, but also of the motion of deformable solids in close interaction with the physico-chemical processes occurring within a given particle and those attending its interaction with the neighboring particles of the body and with external objects.

The world literature of recent years contains numerous theoretical papers in which new types of generalized forces and equations of state are introduced. Most of these studies are based on formal mathematical constructions.

The construction of new theories is intimately connected with the introduction of new concepts as defining and unknown characteristics. It also involves quantities which are defined mathematically to describe the properties of space and time, the positions and states of substantive body particles and fields. These new concepts and mathematical entities make it possible to isolate the defining quantities from the general laws of motion and physico-chemical processes.

To consider these matters more specifically, let us examine the general formulation of problems of constructing models to describe broad classes of motions and processes in the mechanics of continuous media.

Let us begin with some examples of basic characteristic quantities.

Physical investigation of the motion of material continua entails the use of the concepts of time and of a three- or four-dimensional metric space; it always requires two coordinate systems (see Fig.1)(**), namely the observer's coordinate system x^1, x^2, x^3, x^4

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*) The present paper was delivered at the opening session of the Third All-Union Congress on Mechanics (January 25, 1968).

**) Some authors hold the view that the mechanics of movable continuous material media can be constructed by means of a single Cartesian coordinate system without significantly limiting generality. This supposition, which is reflected in certain texts and conveyed to students in all sincerity by their teachers, is incorrect and hinders proper understanding of mechanics and its problems. Confusion is bred, on the one hand, by the fact that the mechanics of deformable bodies is usually concerned with linear problems in which one can assume the the observer's system coincides with the comoving system. On the other hand, it is encouraged by the fact that the metric of the comoving Lagrangian coordinate system in the theory of liquids and gases is manifested only by way of density. At the same time it is often forgotten that even though all substantive characteristics such a velocity, acceleration, strain rate tensor, etc., are introduced by way of and the corresponding Lagrangian system ξ^1, ξ^2, ξ^3 $\xi^4 = t$. In Newtonian physics we can always assume that $x^4 = \xi^4 = t$ and consider absolute time as a scalar variable. The coordinates ξ^1, ξ^2, ξ^3 define the positions of individual particles. In general, both coordinate systems are curvilinear.

A length element in a metric Riemannian space is given by

$$ds^2 = g_{ij}\, dx^i dx^j = \hat{g}_{ij}\, d\xi^i d\xi^j \tag{1}$$

The components of the tensor g_{ij} define the metric and are the basic characteristics of space and time.

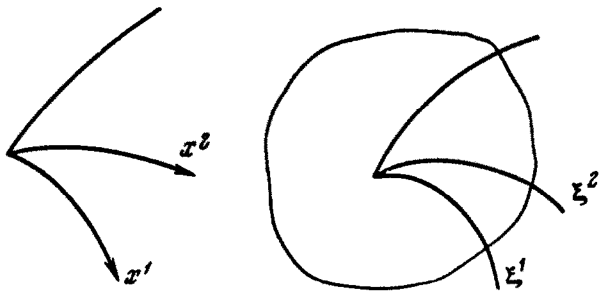

Fig. 1

In Newtonian mechanics and the special relativity theory the tensor g_{ij} is Euclidean, and the definitions of its components are supplemented by the observer at his own discretion solely through his choice of the coordinate system x^1, x^2, x^3, x^4.

In the general theory of relativity the tensor g_{ij} is determined from equations expressing physical principles. The invariant differential quantities which define the properties of the metric tensor g_{ij} of a four-dimensional Riemannian space can be taken as the first and very important example of nonclassical physical unknowns of a new type.

The basic unknown relationship in the observer's system which defines the motion of the medium is the law of motion represented by the four functions

$$x^i = x^i(\xi^1,\ \xi^2,\ \xi^3,\ \xi^4) \qquad (i = 1,2,3,4) \tag{2}$$

In addition to the functions $x^i(\xi^k)$ it is convénient to introduce the following derivatives as defining arguments for various physical functions:

$$x_j{}^i = \frac{\partial x^i}{\partial \xi^j},\quad \nabla_{k_1} x_j{}^i, \ldots \nabla_{k_1}\nabla_{k_2}\cdots\nabla_p \overset{i}{x_j}\cdots \qquad (p = 1,\ 2,\ 3,\ \ldots) \tag{3}$$

Here the symbol ∇_k denotes a covariant derivative with respect to x^k; the first derivatives $x_j{}^i$ can be regarded for fixed values of the subscript j as vector components over the index i. These vectors define the components of the velocity vector, the corresponding rotations, and the components of the strain tensor

$$\hat{\varepsilon}_{ij} = {}^1\!/_2\,(\hat{g}_{ij} - g^\circ_{ij}) = {}^1\!/_2\,(g_{pq}\, x_i{}^p x_j{}^q - g^\circ_{ij})$$

in the comparison of a given position of the body with some imagined "initial position".

Here $g_{ij}{}^\circ(\xi^1,\ \xi^2,\ \xi^3,\ \xi^4)$ denote the components of the metric tensor corresponding to the "initial position" which is introduced by some convention based on physical considerations. In the simplest special cases the initial position is introduced as an "unchanging solid body" whose three-dimensional spatial part coincides with the given deformable body at some "initial" instant (see [1]).

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the observer's coordinate system, the notion of the comoving coordinate system is still essentially involved.

Together with law of motion (2) we must introduce the variable parameters μ^A and their gradients (covariant derivatives) of various orders (4)

$$\mu^A = \mu^A(\xi^1, \xi^2, \xi^3, \xi^4), \quad \nabla_{k_1}\nabla_{k_2}\cdots\nabla_{k_q}\mu^A, \ldots \quad (A = 1, 2, \ldots, N;\ q = 1, 2, 3, \ldots)$$

For our additional parameters μ^A we can take:

the entropy and the concentrations of various components in the mixture;

the components of the tensors of residual strains and dislocation density(*) $\varepsilon_{ij}^{(p)}$, S_{ij},

the components of the electromagnetic potential vector A_i for the electromagnetic field tensor,

$$F_{ij} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$$

defined in the appropriate inertial coordinate system by the matrix (e.g. see [3])

$$F_{ij} = \begin{Vmatrix} 0 & B^3 & -B^2 & cE_1 \\ -B^3 & 0 & B^1 & cE_2 \\ B^2 & -B^1 & 0 & cE_3 \\ -cE_1 & -cE_2 & -cE_3 & 0 \end{Vmatrix}$$

where c is the velocity of light, E_1, E_2, E_3 are the components of the electric intensity vector, and B^1, B^2, B^3 are the components of the magnetic induction vector;

the components of the magnetization and polarization tensor $\mathcal{P}_{ij} = {}^1/_2 (F_{ij} - H_{ij})$, where the H^{ij} are given by the matrix

$$H^{ij} = \begin{Vmatrix} 0 & H_3 & -H_2 & -D_1/c \\ -H_3 & 0 & H_1 & -D_2/c \\ H_2 & -H_1 & 0 & -D_3/c \\ D_1/c & D_2/c & D_3/c & 0 \end{Vmatrix}$$

where H_1, H_2, H_3 are the components of the magnetic intensity vector, and D_1, D_2, D_3 are the components of the electric induction vector;

the components of the internal mechanical moments of momenta m_{ik}, etc.

The variable parameters μ^A can be scalars, tensors, or spinors [4, 5 and 6]. The presence of the variable parameters μ^A which must be determined (see (4)) in the solution of problems means that the model of a continuous medium under consideration has internal degrees of freedom.

A characteristic and important feature of all macroscopic models of deformable media and fields is the functional dependence of the unknown quantities for bodies of finite dimensions on the defining parameters. For example, for a deformable body of finite dimensions, the total internal energy U is always a functional of the functions $x^i(\xi^k)$ and $\mu^A(\xi^k)$.

In many practical cases it is possible to make use of the generalized property of additivity of the internal energy and to express the total energy U in the form (5)

$$U = \int_m u(g_{ij}, x_j^i, \ldots \nabla_{k_1}\nabla_{k_2}\cdots\nabla_{k_r}x_j^i, \mu^A, \ldots \nabla_{k_1}\cdots\nabla_{k_s}\mu^A, S, K_B)\, dm + U_0$$

where m is the rest mass, dm a rest mass element of the medium, and u the local internal energy per unit mass (a physically defined function of the indicated arguments only);

*) The theory of dislocations is presently being developed by refinement and generalization of plasticity theory through the addition of new parameters (e.g. see [2]).

S is the entropy, and K_B ($B = 1,2,...$) are known functions of the coordinates ξ^i (a generalization of the specified physical constants). By fundamental physical hypothesis, the total specific energy u at a given point does not depend on higher-order gradients(*) not present among the arguments of u (r and s are fixed numbers).

In the classical elasticity theory we have the simplest case where

$$u\,(\overset{\circ}{g}_{ij}, \varepsilon_{ij}, S, K_B)$$

In the more complex new models (**) of continuous media the arguments of the specific internal energy u also contain the additional physico-chemical characteristics μ^A and gradients of various orders of the quantities $x_j{}^i$ and μ^A.

The presence of such gradients in the expression for the internal energy makes it necessary to reconsider our concepts concerning the equations of motion and processes, boundary and initial values, interaction mechanisms, conditions at discontinuities, and many other matters.

The constant U_0 specially isolated and emphasized in Formula (5) is enirely immaterial in classical elasticity theory and is usually set equal to zero.

In the more general case the constant U_0 must be allowed for and cannot be regarded as an additive quantity for the individual parts of the body when the body is, in fact, divided into separate parts.

This is because any separation of a body into parts, any fragmentation, etc., involves losses of external energy.

In the first approximation the nonadditivity of the total internal energy U can be allowed for by means of the constant U_0. Allowance for the variation of U_0 with changes in the body surface due to cracking, appearance and development of dislocations, and destruction is of paramount importance.

For elastic bodies with isolated singularities it is possible to find changes in the constant U_0 for equilibrium processes from the total changes in elastic energy. The production or elimination of certain defects in the body through the action of internal processes or certain external influences requires energy whose sources may be the total internal energy of the body and known external energy inputs. In some cases changes in U_0 are analogous to latent heat of fusion, or to phase-change energy in general.

It must be noted that further investigation of the strength of materials on a physical basis will be closely related to the analysis of changes in U_0. The lack of finished theories and notable successes in the solution of problems on materials strength criteria can be attributed to disregard of the quantity U_0. At the same time, advances in the theory of cracking of brittle bodies have been due largely to allowance for changes in U_0.

In solutions of certain problems arrived at within the framework of elasticity theory, the theoretical stresses in certain small domains can increase without limit without noticeable or even local fracture. Because of this, fracture criteria based on the appearance of theoretical stresses in excess of limiting values in an elastic field are sometimes

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*) The possibility of having higher-order derivatives among the arguments of prescribed functions was already foreseen and predicted by Cauchy when he laid the groudwork of elasticity theory. Limiting transitions from a discontinuum to the continuum in statistical theories indicate that the arguments of the specific internal energy u can generally include derivatives (3) of any order.

**) For example, the model of a bubble-containing liquid in [7].

inadequate.

Fracture of various structural components and test specimens is generally a global phenomenon of the same character as motion instability, impossibility of equilibrium, or impossibility of continuous motion.

Fracture criteria are generally nonlocal in character. Nevertheless, global instability is often determined by entirely local conditions. One must bear in mind, however, that in many cases the corresponding local conditions may only be necessary, and not sufficient for loss of stability of equilibrium and for fracture of a given structure.

The problem of constructing models of continuous media consists in identifying the characteristic quantities and constructing a system of functional or differential equations and various additional conditions which make it possible to formulate mathematical problems of determining the laws of motion $x^i(\xi^k)$ and the physico-chemical processes defined by the functions $\mu^A(\xi^k)$ for specific physical situations.

The problem of constructing models of continuous media for known classes of real objects and real phenomena is one of the basic problems of physics. Solution of this problem must be founded on universal and particular basic assumptions, on experimental data, and of the correlation of observations and experimental measurements with theoretical conclusions and computations within the limits of accuracy required in practice or implied by the meaning of a given problem.

The present paper contains a description, analysis, and elaboration of the general method which makes it possible to obtain complicated closed systems of equations and complicated supplementary boundary and other conditions for models of media with internal degrees of freedom from the minimum number of physical assumptions. The additional boundary and other conditions just mentioned are a means of rendering specific ("concretizing") individual models and particular formulations of problems.

The basic variational equation which we propose to investigate and which constitutes the foundation of the present treatise is a simple and natural generalization of the variational principle of Lagrange. In many highly important cases it coincides completely with the familiar applications and formulations of this principle [8, 4, 6, 9, 1 and 10].

As we have known for a long time, all of the basic equations of relativity theory, electrodynamics, analytical mechanics, thermodynamics of equilibrium processes, elasticity theory, hydrodynamics, and many other disciplines result from the application of the Lagrange variational principle.

In many modern physical theories this variational principle constitutes a working and essentially unique initial investigative apparatus.

Our analysis will show that the Lagrange variational equation for material continua and physical fields can be employed as a basis for all physical models not only of reversible phenomena, but in cases of irreversible phenomena as well.

The variational equation has made it possible to unify and synthesize on a common basis various phenomenological and statistical methods of the theory of irreversible processes in thermodynamics and mechanics. In particular, it has permitted the interpretation and evaluation of the associated law of residual plastic strains in mechanical plasticity theory within the framework of the existing thermodynamics of irreversible processes.

A new element of the theory which we shall develop will be the use of the variational equation for:

1) describing irreversible phenomena realizable in continuous media;

2) establishing equations of state;
3) establishing kinetic equations;
4) obtaining initial and boundary conditions;
5) obtaining conditions at strong discontinuities ("jumps") inside a medium.

In elaborating the modern theory of complicated macroscopic models of media and fields it is important to bear in mind that even in Newtonian mechanics the description of phenomena with significant involvement of internal degrees of freedom on the basis of only the principal equation of Newtonian mechanics

$$ma = \mathbf{F} \tag{6}$$

is impossible.

Eq. (6) is a sufficient basis for developing the analytical mechanics of a system of material points, the theory of absolutely solid bodies, adiabatic elasticity theory, and the theory of motion of an ideal incompressible fluid, and certain other disciplines. It is already inadequate, however, for considering macroscopic thermal and electromagnetic effects.

Specifically, Eq. (6) cannot serve as a basis for obtaining the macroscopic laws governing the growth of plastic strains, for consideration of effects associated with the variation of continuously distributed dislocations, for taking account of various processes and effects associated with macroscopic theories of electric polarization and magnetization of media, and for many other purposes.

For example, the familiar equation for the moments of momenta for small particles or finite bodies does not follow from Eq. (6), but is rather an independent fundamental equation derivable from the symmetry of natural laws relative to a rotation group. Eq. (6) is, in fact, a consequence of the symmetry of natural laws relative to a translation group.

For an absolutely solid body and for many classical models of continuous media the differential equation of the moments of momenta reduces to the condition of symmetry of the internal stress tensor or is satisfied automatically when the internal stress tensor is introduced as a characteristic to be determined from the general assumptions concerning the properties of the medium.

We note that the development of statistical theories for the derivation of macroscopic relations on the basis of Eq. (6) on the macroscopic level always involves some additional universal and particular assumptions which do not follow directly from Eq. (6).

Now let us consider the meaning of the basic variational equation which can be considered as the fundamental point of departure for macroscopic media with internal degrees of freedom.

For simplicity and greater generality we shall carry out our discussion within the framework of special relativity theory, assuming that space-time is pseudo-Euclidean.

Experience and close examination indicate the development of the theory using a four-dimensional geometrically defined physical space-time and four-dimensional vectors and tensors is very convenient, natural, and quite necessary from the physical standpoint in certain cases.

In the observer's fixed coordinate system we mentally complement the real motions and processes described exactly or approximately by means of the piecewise-continuous functions

$$x^i(\xi^k), \qquad \mu^A(\xi^k), \qquad S(\xi^k) \tag{7}$$

by some sufficiently broad class of piecewise-continuous permissible functions (which by hypothesis contains system of functions (7))

$$\tilde{x}^i(\xi^k) = x^i(\xi^k) + \delta x^i, \qquad \tilde{\mu}^A(\xi^k) = \mu^A(\xi^k) + \delta\mu^A, \qquad \tilde{S}(\xi^k) = S(\xi^k) + \delta S \tag{8}$$

and, in view of the meaning of the quantities $K_B(\xi^k)$, assume that

$$\delta K_B(\xi^k) = 0$$

The functions $\tilde{x}_i$, $\tilde{\mu}^A$, $\tilde{S}$ are considered at points of some domain of a system of events of the four-dimensional volume V_0 in space-time bounded by the three-dimensional surface Σ_0. The construction to follow involves the assumption that in the class of permissible functions the variations δx^i, $\delta\mu^A$ and δS in the volume V_0 are continuous together with all their derivatives entering into the variational equations and that they are sufficiently arbitrary, while the variations δx_j^i, $\delta\nabla_k x^i{}_j,\ldots$, $\delta\nabla_k\mu^A,\ldots$ etc., can be expressed in terms of the functions $x^i(\xi^k)$ and $\mu^A(\xi^k)$ for real phenomena, in terms of the variations δx^i and $\delta\mu^A$, and in terms of their derivatives with respect to the coordinates x^i.

The following are important new features of the theory we are developing:

1) the variations δx^i are defined as components of a four-dimensional contravariant vector, and the variations $\delta\mu^A$ are defined as components of tensors of the same species as μ^A;

2) the variations δx^i and $\delta\mu^A$ and their derivatives can be different from zero and are to some extent arbitrary on the boundaries Σ_3 of the arbitrary volumes $V_4 < V_0$

We write the fundamental basis equation in the form

$$\delta \int\limits_{V_4} \Lambda d\tau + \delta W^* + \delta W = 0 \tag{9}$$

where Λ is the density of the Lagrangian.

For a material medium we can express Λ as (*)

$$\Lambda = -\rho u\,(g_{ij},\ x_j^i,\ \nabla_k x_j^i \ldots \mu^A,\ \nabla_k\mu^A, \ldots,\ S,\ K_B) \tag{10}$$

where ρ is the scalar density (the ratio of the rest mass to the three-dimensional volume in the comoving coordinate system), and u is the internal energy per unit rest mass in the comoving coordinate system. In special relativity theory the quantity u can be considered as a four-dimensional scalar. The first law of thermodynamics says that the function $u\rho d\tau$ can be introduced for any infinitely small physical particle.

Determination of the arguments and the form of the function u is the basic physical problem arising in the "concretization" of a model of a continuous medium. Stipulation of the internal energy as a function of its arguments always involves certain assumptions, some of which may appear very natural and self-evident.

In practice the values of variable parameters can often be considered as characteristics

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*) To the variable integral over V_4 we can add terms allowing for the presence of the quantity U_0 in Formula (5), which can generally vary due to the development of the boundary Σ_0 and of the discontinuity surfaces inside V_4. Such an additional term is not introduced in the basic variant of the theory presented below. In the arguments of the formula for Λ we have isolated the entropy S from among the parameters μ^A. The arguments of the formula for Λ do not include gradients of the entropy S. The subsequent theory can be extended directly to the case where the entropy is not specially isolated, but is rather identified with one of the parameters μ^A entering into Λ together with its gradients of any order.

of small perturbations. For this reason the function u can be considered simply as a positive-definite form of the defining small variable parameters. In this case the problem of determining the function u reduces to the problem of determining the constant coefficients of the corresponding quadratic form. Determination of these coefficients is made easier by symmetry conditions [11 and 12], and can be based on experimental data. In certain cases the values of these coefficients can be related to molecular constants on the basis of a statistical theory (developed with its universal assumptions and assumptions specific to the given model). Such coefficients are similar to Young's modulus and to the Poisson coefficient which, in practice, can always be readily found by experiment. They can be computed statistically (on the basis of certain far-reaching assumptions). However, in the case of certain solids the statistical values do not agree in general with those obtained experimentally. Agreement between theory and experiment is better for gases, but here too experimental verification of theoretical results is necessary. Nevertheless, statistical theories provide a means of gaining insight into certain relations between such coefficients which are not clear from phenomenological theories, i.e. into the relationship among the coefficients of heat conduction, viscosity, and diffusion.

In an internal coordinate system in Newtonian mechanics Formula (10) can generally be replaced by Formula

$$\Lambda = \rho\,(^1/_2 v^2 - u)$$

where v is the velocity of the points of the continuous medium and u is a three-dimensional scalar equal to the internal energy.

In the theories already developed the function Λ can be considered known both for models of material media already defined and for the electromagnetic field. In the general theory of relativity the additive component of the quantity Λ associated with gravity is known and serves as a basis for determining the metric tensor g_{ij} representing the gravitational field. The various generalizations of general relativity theory, generally speaking, always involve a change in, or some other specification of, the density of the Lagrangian Λ.

It is important to note that from the physical standpoint we can say that a physical system has been specified or is known only if the internal energy or, respectively, the Lagrangian Λ, has been specified or determined [1, 10 and 13 – 15].

Thus, the requirement of specifying the Lagrangian Λ as a function of macroscopic variables in Eq. (9) is a natural one from the physical standpoint. In satisfying this requirement we can draw on the immense body of experience accumulated in the various branches of physics and in various experiments. The assumptions made in specifying the function Λ are always necessary and can be justified by various intuitive and other, generally simpler, assumptions.

In discussing the problem of specifying the function Λ one can and must establish the most intimate possible contact between macroscopic theory, universal physical principles, experiment, and statistical theories.

Now let us examine the expression for the specified functional δW^* characterizing the external volume interactions (in V_4) and surface interactions (on Σ_3) of a given portion of the medium in V_4 with external fields and bodies, and for certain irreversible actions of neighboring parts of the medium adjacent to the isolated volume V_4 along the surface Σ_3.

For adiabatic reversible processes in the absence of external energy influxes inside V_4 and on the surface Σ_3 it is often possible to assume simply that

$$\delta W^* = 0$$

In conservative systems of celestial mechanics we can always assume that $\delta W^* = 0$.

In the general case of phenomenological theories where there are volume and surface energy influxes external to the medium under consideration and where irreversible processes operating with the result that the arguments of Λ include derivatives of various orders of $x^i(\xi^k)$ and $\mu^A(\xi^k)$ with respect to ξ^k or x^i, we can write the following general expression for δW^*:

$$\delta W^* = \int_{V_4} (\rho\theta\delta S - Q_i\delta x^i - M_A\delta\mu^A)\, d\tau - \tag{11}$$

$$- \int_{\Sigma_3 + S_\pm} \left(\sum_{p=0} Q_i^{kj_1 \ldots j_p} \nabla_{j_1} \ldots \nabla_{j_p} \delta x^i + \sum_{q=0} M_A^{kj_1 \ldots j_q} \nabla_{j_1} \ldots \nabla_{j_q} \delta\mu^A \right) n_k d\sigma$$

Here $S_\pm$ denotes the two sides of the three-dimensional surface S inside V_4 at which the characteristics of motion can experience strong discontinuities; n_k are the components of the unit vector of the exterior normal to Σ_3 and S_+ or S_-. The components

$$Q\,(Q_i,\, Q_i^{kj_1 \ldots j_p}),\; M\,(M_A,\, M_A^{kj_1 \ldots j_q})$$

are some prescribed external generalized "forces". The quantity θ plays the role of the absolute temperature, and can be regarded either as an unknown or as a prescribed quantity, depending on the circumstances. The entropy variation δS in Formula (11) was introduced as a quantity independent of the variations δx^i and $\delta\mu^A$.

Specification of the functional δW^* involves the problem of discriminating between internal and external interactions. For example, if the electromagnetic or gravitational field is considered as an external object, then the corresponding energy influxes for the electromagnetic ponderomotive forces and gravitational forces are present in the expression for δW^*; on the other hand, if these fields are included in the model of the medium, then the corresponding total differentials are separable from δW^* and must be included in the expression for Λ. Upon transfer of the total differentials from δW^* into $\int \Lambda d\tau$ the meaning of Λ changes, and Formula (10) can be replaced by another similar one which contains free energy or enthalpy instead of the internal energy, and which may contain other thermodynamic functions of state. With irreversible processes transfer of the complete term δW^* into Λ is impossible, since the variation δW^* is generally nonholonomic.

Definition of the components of the generalized mass and surface forces Q and M is a problem closely related to the theory of dissipative mechanisms. Solution of this problem necessarily entails various assumptions and contacts with the existing thermodynamics of irreversible phenomena. Detetermination of Q and M is analogous to the basic physical problem of Newtonian mechanics on the determination of laws for forces defined by Newton's equation, and in our case by variational equation (9). Consideration of the properties of the quantities appearing in the integrand of the expression for δW^* at the discontinuity surface $S_\pm$ can have special physical significance. Determination of δW^* involves choosing the defining parameters x^i and μ^A, determining their variations, and considering the property of continuity of the variations at discontinuities.

It is important to note that the determination or specification of the quantities Λ and δW^* serves to delineate common bases for the most varied models. This makes possible the use and synthesis of experience in various disciplines and the establishment of direct correlations between different theories. Moreover, additional means of using statistical considerations arise.

Dissipative processes must and can be conveniently allowed for by means of the entropy production equation in the laws of real motions and processes. The equation describing the variation of particle entropy is derived below from the Euler equations in general variational problem (9). The positiveness of entropy growth due to irreversible internal processes must be ensured by the laws defining Λ and the generalized forces Q and M for real phenomena.

In accordance with the basic meaning of Eq. (9) we assume that the quantity δW is a surface integral over $\Sigma_3 + S_\pm$. For variations δx^i, $\delta\mu^A$ and their derivatives not equal to zero on $\Sigma + S_\pm$, the variation δW can be determined from Eq. (9) in terms of $\delta\int\Lambda d\tau$ and δW^*.

If the quantity δW is defined on $\Sigma_0 + S_\pm$ not only by Eq. (9) (for arbitrary δx^i, $\delta\mu^A$ and their respective derivatives), but also by external conditions, then, as we shall show below, this yields initial conditions, boundary conditions, and conditions at the discontinuity.

With the variations δx^i and $\delta\mu^A$ and their derivatives of proper order equal to zero on $\Sigma_3 + S_\pm$ but arbitrary (linearly independent) inside V_4 Eq. (9) yields the Euler equations (*)

$$\frac{\delta\Lambda}{\delta x_q^p}\nabla_i x_q^p + \nabla_s\left(\frac{\delta\Lambda}{\delta x_q^i}x_q^s\right) + \frac{\partial\Lambda}{\partial k_B}\nabla_i K_B + Q_i + M_A\nabla_i\mu^A = \rho\theta\nabla_i S \quad (12)$$

$$\frac{\delta\Lambda}{\delta S} = \frac{\partial\Lambda}{\partial S} = -\rho\theta, \qquad \frac{\delta\Lambda}{\partial\mu^A} = M_A \quad (13)$$

Here $\delta\Lambda/\delta x_q^p$, $\delta\Lambda/\delta\mu^A$ and $\delta\Lambda/\delta S$ denote variational derivatives, e.g.

$$\frac{\delta\Lambda}{\delta x_q^p} = \frac{\partial\Lambda}{\partial x_q^p} - \nabla_k\frac{\partial\Lambda}{\partial\nabla_k x_q^p} + \nabla_k\nabla_s\frac{\partial\Lambda}{\nabla_s\nabla_k x_q^p} - \dots \quad (14)$$

Multiplying Eq. (12) by x_4^i and summing over the index i, we obtain

$$\rho\theta\frac{dS}{d\xi^4} = Q_i\frac{\partial x^i}{\partial\xi^4} + M_A\frac{d\mu^A}{d\xi^4} + \frac{\partial\Lambda}{\partial K_B}\frac{dK_B}{d\xi^4} + \nabla_s F^s \quad (15)$$

where

$$F^s = x_4^i x_p^s\frac{\delta\Lambda}{\delta x_p^i}, \qquad \frac{d}{d\xi^4} = \frac{\partial x^i}{\partial\xi^4}\nabla_i$$

Since

$$x_4^s\frac{\delta\Lambda}{\delta x_q^p}\nabla_s x_q^p + x_4^p\nabla_s\left(\frac{\delta\Lambda}{\delta x_q^p}x_q^s\right) = \nabla_s\left(x_4^i x_q^s\frac{\delta\Lambda}{\delta x_q^i}\right)$$

by virtue of Eq.

$$0 = \frac{\partial\Lambda}{\partial x_q^p}(x_4^s\nabla_s x_q^p - x_q^s\nabla_s x_4^p) = \frac{\partial\Lambda}{\partial x_q^p}\left(\frac{\partial^2 x^p}{\partial\xi^q\partial\xi^4} - \frac{\partial^2 x^p}{\partial\xi^4\partial\xi^q}\right)$$

Eq. (15) is the equation for entropy production in a particle, since, by hypothesis, the coordinate ξ^4 plays the role of time. To obtain the derivatives with respect to proper time $d\tau = (g_{\hat{4}\hat{4}})^{1/2}d\xi^4$ we need merely multiply both sides of relation (15) by $(g_{\hat{4}\hat{4}})^{-1/2}$.

The Euler equations contain the impulse and energy equations. Depending on the meaning of the parameters μ^A, the Euler equations also contain the Maxwell equations, chemical kinetics equations, and various other forms of equations for the required parameters μ^A characterizing the internal degrees of freedom. It can be shown [2] that all

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*) These equations were obtained by equating the coefficients of $\partial x^i = \delta x^i$, $\partial\mu^A$ and δS in the volume integral to zero, taking account of Eqs.

$$\delta\Lambda = \partial\Lambda + \delta x^i\nabla_i\Lambda, \quad \delta\mu^A = \partial\mu^A + \delta x^i\nabla_i\mu^A, \quad \delta d\tau = \nabla_i\delta x^i d\tau$$

existing macroscopic models of continuous media, including models of plastic media, can be obtained from basis equation (9).

The Euler equations are generally partial differential equations whose order is related to the order of the derivatives entering into the arguments of the Lagrangian Λ. In the general case this order can be fairly high.

If $\Lambda d\tau$ and δW^* are four-dimensional scalars defined by Formulas (10) and (11), then after variation of the first integral and appropriate integration by parts, basic Eq. (9) yields Formula

$$\delta W = \int_{\Sigma_s + S_\pm} \Big[\sum_{p=0} (\mathrm{P}_i^{kj_1\ldots j_p} + Q_i^{kj_1\ldots j_p}) \nabla_{j_1} \ldots \nabla_{j_p} \delta x^i + \tag{16}$$

$$+ \sum_{q=0} (N_A^{kj_1\ldots j_q} + M_A^{kj_1\ldots j_q}) \nabla_{j_1} \ldots \nabla_{j_q} \delta\mu^A \Big] n_k d\sigma + \int_{\Sigma_s + S_\pm} \nabla_s \Omega^{sk} n_k d\sigma$$

Here $\mathrm{P}_i^{kj_1\ldots j_p}$ and $N_A^{kj_1\ldots j_q}$ are certain quantities (tensor components) expressible in terms of Λ and derivatives of x^i and μ^A. These quantities result upon transformation of the variation

$$\delta \int_{V_4} \Lambda d\tau$$

by integration by parts. These transformations do not yield unambiguous definitions of the components $P_i^{kj_1\ldots j_p}$ and $N_A^{kj_1\ldots j_q}$. This is because of the possibility of adding the final integral to the left side of (16). This integral is equal to zero when Ω^{sk} is an arbitrary antisymmetric tensor with discontinuous components having continuous first- and second-order derivatives at the points of the volume bounded by the surface $\Sigma_3 + S_\pm$.

This statement is a self-evident consequence of the Gauss-Ostrogradskii theorem, since Eq. $\Omega^{sk} = -\Omega^{sk}$ implies that $\nabla_s \nabla_k \Omega^{sk} = 0$.

Any linear forms of the same character as those in the first terms of the integrand in Formula (16) can be taken as the components of Ω^{sk}. It is clear that the formulas which yield expressions for the tensor components

$$\mathrm{P}_i^{kj_1\ldots j_p} + Q_i^{kj_1\ldots j_p}, \qquad N_A^{kj_1\ldots j_q} + M_A^{kj_1\ldots j_q}$$

in terms of parameters characterizing the motion and state of the particles are not uniquely defined because of the arbitrary choice of Ω^{sk}.

This gives rise to the question of the ambiguity of the notion of the energy-momentum tensor, as well as to the question of arbitrariness for specified Euler equations, for equations of state in general, and for the fundamental notion of internal stresses in particular.

The dependence of the indicated tensor components in Formula (16) on the defining parameters can be regarded and interpreted as the equations of state of the physical medium. These equations constitute a generalization of Hooke's law.

Thus, arbitrariness in defining the equations of state arises for a specified system of Euler equations. More detailed analysis shows that additional boundary and initial conditions at the strong discontinuities which express physical interactions at the boundary of the body or at discontinuities inside the body do not constitute a basis for eliminating the above ambiguity of the equations of state.

For a specified system of Euler equations it is possible to alter the density of the Lagrangian Λ by adding a divergent term. It is clear that this implies a change in the equations of state. However, complete specification of the Lagrangian can be incorporated into the physical definition of a model of a continuous medium. Specification of the system of

Euler equations does not provide the complete and necessary information about a specific model of a medium.

The stresses are, of course, defined unambiguously once the equations of state have been established. But the whole significance of the ambiguity under discussion has to do with the fact that all the laws of motions and laws of variation of the parameters μ^A in specific problems remain valid for certain other forms of the equations of state.

We emphasize that the ambiguity under discussion is not related to the specifics of the method used to establish the equations of state using variational principle (9). The same situation arises in using the general heat influx equation of thermodynamics in differential form [14].

The significance of the ambiguity can be undestood and explained on the basis of the following physical considerations.

It is a well know fact that the problem of internal stresses in an absolutely rigid body in motion has no definite solution. It is always possible to imagine any system of internal forces equivalent to zero in such a body without being able to detect its presence or absence. The equations of motion and additional conditions for various systems of internal stresses are equal, while their equations of state differ.

It is clear that this ambiguity does not arise when the equations of state are prescribed. However, in the construction of new models, i. e. when the system of equations of state is being established, the possibility of choosing different equations of state arises by the nature of the problem. This can assume considerable importance when the density of the Lagrangian depends on the sequence of gradients defining characteristics.

In order to illustrate the validity of this statement, let us consider the equations of elasticity theory for which the equations of state are given by Formulas

$$p^{ij} = \rho(\partial u/\partial \varepsilon_{ij}) \tag{17}$$

Instead of equation of state (17) we choose other equations of state of the form

$$p^{*ij} = p^{ij} + \widetilde{p}^{ij}, \qquad \widetilde{p}^{ij} = \nabla_s \nabla_k N^{iksj} \quad (N^{iksj} = -N^{ikjs}) \tag{18}$$

where the quantities N^{iksj} are, as indicated, antisymmetric in s and j, i. e. they constitute components of a tensor which in all problems depends in the same but arbitrarily specified way on any of the parametets of state and any of their derivatives (*).

It is clear that all the laws of motion and straining will be defined independently of $\widetilde{p}^{ij}$ since the additional stresses $\widetilde{p}^{ij}$ identically satisfy the equations of equilibrium

$$\nabla_j \widetilde{p}^{ij} = 0$$

and since, moreover, for the volume V bounded by a finitely closed surface Σ we have

$$\int_V \nabla_i \widetilde{p}^{ij} \delta x_i d\tau = \int_\Sigma (\widetilde{p}^{ij}\delta x_i + \nabla_k N^{iksj} \nabla_s \delta x_i) n_j \, d\sigma = \int_\Sigma \nabla_s (\nabla_k N^{iksj} \delta x_i) \, n_j d\sigma = 0$$

where the normal components of the gradient ∇_s which appears in the integrand of the surface integral vanish identically at each point of the surface Σ. This implies that the additional stresses $\widetilde{p}^{ij}$ do not contribute to the energy influxes of the interactions (with

*) In elasticity theory problems of equilibrium in the absence of external body forces the solutions of the stress problems can also be represented in the form (18). If Hooke's law or some other specific equation of state applies, however, the quantities N^{iksj} are functions of the coordinates and not universal functions (the same for all problems) of the strain characteristics.

arbitrary possible displacements δx_i) between neighboring particles of any surface Σ, and therefore with external bodies at the boundary Σ_0 of the body.

Variational equation (9) affords deeper insight into the concepts of equations of state, boundary and initial conditions, and conditions at strong discontinuities which do not follow from the differential equations without additional assumptions. It turns out that all of the conditions and equations just listed are interrelated and must be considered as a unified whole.

The conclusions to follow are related to a transformation of Formula (16) for δW such that the integrand contains only the variations δx^i and $\delta\mu^A$ and the covariant derivatives along the normal $\nabla_n^{(\alpha)}\delta x^i$ and $\nabla_n^{(\beta)}\delta\mu^A$ independent on $\Sigma + S_\pm$. Here $\alpha, \beta = 1,2,...$ The fact is that the variations δx^i and $\nabla_j \delta x^i$ and not all of the higher-order gradients $\nabla_{j_1},...,\nabla_{j_p}\delta x^i$ on $\Sigma + S_\pm$ can be considered independent.

In the simplest particular cases the appropriate transformations of Formula (16) for obtaining boundary conditions were carried out by Mindlin (*) [17]. The appropriate particular transformations for obtaining the conditions at discontinuities were developed by Lur'e [18].

Let us assume that the surface $\Sigma_3 + S_\pm$ is smooth. A sufficient condition for this is that the surface S be smooth (since the volume V_4 and the chosen surface Σ_3 are arbitrary). The above transformations yield Formula

$$\delta W = \int\limits_{\Sigma + S_\pm} (\mathscr{P}_{i0}\delta x^i + \mathscr{P}_{i1}\nabla_n \delta x^i + \ldots + \mathscr{P}_{i(r-1)}\nabla_n^{(r-1)}\delta x^i + \\ + \mathscr{M}_{A0}\delta\mu^A + \mathscr{M}_{A1}\nabla_n\delta\mu^A + \ldots + \mathscr{M}_{A(s-1)}\nabla_n^{(s-1)}\delta\mu^A)\, d\sigma \tag{19}$$

In Formula (19) the components of the vectors $\mathscr{P}_{i0}, \mathscr{P}_{i1},..., \mathscr{P}_{i(r-1)}$ and the components of the tensors $\mathscr{M}_{A0},...,\mathscr{M}_{A(s-1)}$ are defined uniquely and are expressed in terms of $P_i^{kj_1,...,j_\nu} + Q_i^{kj_1,...,j_\nu}$ and $N_A^{kj_1,...,j_\nu} + M_A^{kj_1,...,j_\nu}$ which are not uniquely defined.

An important property of the vectors $\mathscr{P}_\alpha$ and tensors $\mathscr{M}_{A\beta}$ defined at points of elements $d\sigma$ on the boundary surface $\Sigma_3 + S_\pm$ is their dependence not only on the orientation of these elements as in the case of ordinary stresses, but also on the curvature of these elements and other, more suitable, differential-geometric properties of the elements in question (**).

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*) Zhelnorovich carried out the general transformation in four-dimensional space-time for any finite order of the variation gradients.

**) Conversion from Formula (16) to Formula (19) is easily effected in the absence of edges or conic points on $\Sigma + S_\pm$ in the presence of such singularities Formula (19) remains valid, but the value of the integral in (19) must be considered as a limit along the smooth surface $\Sigma + S_\pm$ which tends to a surface with edges. Because the integrand of (19) (which depends on the vector $\mathbf{n}$ and its tangential derivatives) has singularities and discontinuities, taking the limit to a three-dimensional surface $\Sigma + S_\pm$ with two-dimensional edges gives rise to additional integrals taken over the two-dimensional surface with edges. These integrals can be written out by applying integral (16) (which has no singularities) directly to the surface with ribs. It is then necessary to convert to Formula (19); in this transformation the second integral of the divergent term, which vanishes for a smooth surface $\Sigma + S_\pm$, yields a readily computable nonzero integral over the edges in the case of a surface with edges.

The true characteristics of a continuous medium are precisely the vectors $\mathcal{P}_\alpha$ and the tensors $\mathcal{M}_\beta$ which depend on the geometric singularities of the areas on which interaction occurs, and on the defining parameters by way of the Lagrangian Λ and $Q_i^{kj_1,\ldots,j_\nu}$ and $M_A^{kj_1,\ldots,j_\nu}$ which enter into the expression for δW^*. It is clear that the only combinations which matter in Formula (11) for δW^* are those consisting of the $Q_i^{kj_1,\ldots,j_\nu}$ and $M_A^{kj_1,\ldots,j_\nu}$ which enter into the definitions for $\mathcal{P}_{i\alpha}$ and $\mathcal{M}_{A\beta}$.

If the quantity δW is specified on a portion of the boundary Σ_0 then Formula (19), the arbitrariness of δx^i, $\delta\mu^A$, and the normal gradients of Σ_0 imply the following conditions at the points A of the portion of Σ_0 under consideration

$$\mathcal{P}_{i\alpha} = f_{i\alpha}(A), \qquad \mathcal{M}_{A\beta} = g_{A\beta}(A) \tag{20}$$

$$(i = 1,2,3,4; \ A = 1,2,\ldots,N; \ \alpha = 0,1,2,\ldots,r-1; \ \beta = 0,1,2,\ldots,s-1)$$

where $f_{i\alpha}(A)$ and $g_{A\beta}(A)$ are, in general, given functions at the points A.

On the three-dimensional spatial portion of the boundary Σ_0 corresponding to t_0=const Eqs. (20) represent the initial conditions in the three-dimensional volume occupied by the body.

On the three-dimensional portion of Σ formed by the two-dimensional boundary Σ_2 of the body and by the simultaneously varying time t, conditions (20) can be considered as boundary conditions at the boundaries of the variable three-dimensional volume occupied by the given body. Eqs. (20) on the instantaneous boundary $t = \text{const} > t_0$ can generally be considered simply as relations defining the right sides on the basis of the laws of motion isolated by means of the initial and boundary conditions.

Now let us write the conditions of the three-dimensional strong-discontinuity surface S situated inside the four-dimensional volume V_4 of the continuous medium. We assume that on the basis of preliminary studies and appropriate hypotheses all of the external influences on the medium which are distributed over S are included in δW^* (for example, the variation of the "additive" constant u_0, and specifically heat release during chemical reactions at the combustion or detonation front, or else energy absorption at various types of discontinuities along S can sometimes be considered as external influences; the same effects can be interpreted as internal processes due to the complication and variation of the density of the Lagrangian Λ, especially by isolating the variation of the corresponding additional surface integral over the discontinuity surface S).

Assuming that the variations δx^i and $\delta\mu^A$ and all of their derivatives entering into δW are equal to zero on Σ, at the discontinuity surface S, we obtain

$$0 = \delta W = \int_S [(\mathcal{P}_{i0}\delta x^i)_+ + (\mathcal{P}_{i0}\delta x^i)_- + \ldots + (\mathcal{P}_{i(r-1)}\nabla_n^{(r-1)}\delta x^i)_+ +$$
$$+ (\mathcal{P}_{i(r-1)}\nabla_n^{(r-1)}\delta x^i)_- + (\mathcal{M}_{A0}\delta\mu^A)_+ + (\mathcal{M}_{A0}\delta\mu^A)_- + \ldots + (\mathcal{M}_{A(s-1)}\nabla_n^{(s-1)}\delta\mu^A)_+ +$$
$$+ (\mathcal{M}_{A(s-1)}\nabla_n^{(s-1)}\delta\mu^A)]_- d\sigma \tag{21}$$

We assume the same direction of the normal in all quantities in Formula (21) which depend on the direction of the normal to S.

From the definitions of $\mathcal{P}_{i\alpha}$ and $\mathcal{M}_{A\beta}$ and of the operator $\nabla_n{}^k$ we have

$$\mathcal{P}_{i\alpha}(n) = \mp\mathcal{P}_{i\alpha}(-n), \qquad \mathcal{M}_{A\beta}(n) = \mp\mathcal{M}_{A\beta}(-n), \quad \nabla_n^{k-1} = \mp\nabla_{(-n)}^{k-1} \tag{22}$$

where the minus sign corresponds to even, and the plus sign to odd, α, β and k.

As we have already noted, the basic condition of the class of permissible functions consists in the assumption that the required solution and functions under comparison in the volume V_4 are piecewise-continuous, together with all their partial derivatives present in basic variational Eq. (9). The basic significance of introducing the strong-discontinuity surface S inside the volume V_4 lies in the fact that the required solutions and the appropriately varied permissible functions experience discontinuities in the mental intersection of the surface S (*). These discontinuities can be of various types: their character can depend, for example, both on the order and the form of the functions which experience discontinuities on S, or of their derivatives. For example, we can consider strong discontinuities of the crack type in which the required functions together with all of their partial derivatives are discontinuous, or discontinuities of the dislocation type in which the small displacements normal to the surface S are continuous, while the derivatives in the plane tangent to S are discontinuous in passing from one side S_+ to the other side S_-, or discontinuities of the shock-wave type encountered in classical gas dynamics, when all of the coordinates x^i (all of the displacements) on S are continuous, while the derivatives $\partial x^i/d\xi^j$ can experience discontinuities.

When higher-order derivatives $\dfrac{\partial^k x^i}{\partial \xi^{j_1} \cdots \partial \xi^{j_k}}$ are present among the arguments of the function Λ, the number of possible types of strong discontinuities can become quite large.

Two distinct cases are possible in the formulation and solution of specific problems in gas dynamics and in the simple theories of the mechanics of solids: either the type of surface discontinuity is specified, or the type of discontinuity is determined in the course of solution.

Because of this, in using variational equations one is still obliged to introduce or determine classes of functions which must include the required solution (**). In particular, if we assume that the class of permissible functions is defined by the following conditions at points of the surface S:

$$(\nabla_n{}^\alpha \delta x^i)_+ = (\nabla_n{}^\alpha \delta x^i)_- \qquad (i = 1, 2, 3, 4;\ \alpha = 0, 1, \ldots, r_1 - 1;\ \ r_1 \leqslant r) \tag{23}$$

where $(\nabla^\alpha{}_n \delta x^i)_+$, $(\nabla^\alpha{}_n \delta x^i)_-$ are arbitrary and independent for $\alpha = r_1, r_1 + 1, \ldots r - 1$

$$(\nabla_n{}^\beta \delta\mu^A)_+ = (\nabla_n{}^\beta \delta\mu^A)_- \qquad (A = 1, 2, \ldots, N,\quad \beta = 0, 1, \ldots, s_1 - 1,\ s_1 \leqslant s)$$

while $(\nabla_n{}^\beta\, \delta\mu^A)_+$, $(\nabla_n^\beta \delta\mu^A)_-$ are arbitrary and independent for $\beta = s_1, \ldots s - 1$, this defines on passage through the surface S the class of permissible functions $x^j(\xi^1, \xi^2, \xi^3, \xi^4)$ continuous together with their $r_1 - 1$ partial derivatives and the functions $\mu^A(x^1, x^2, x^3, x^4)$ continuous with their $s_1 - 1$ partial derivatives, where the higher-order derivatives of these functions which are normal to S can have an arbitrary discontinuity. In addition to conditions (23) we assume here that all the quantities appearing in Eq. (9) are continuous on each side of the surface S in moving along the surface S. From the arbitrariness and independence of the quantities $\nabla_n{}^\alpha \delta x^j$ and $\nabla^\beta{}_n\ \delta\mu^A$ on the basis of (22) and (23), we obtain from (21) the following conditions at the discontinuity surface:

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*) In general, the magnitudes of the discontinuities of the unknown functions are also unknown. However, there are problems in which some of the discontinuities of the unknowns are specified in additional conditions.

**) Such assumptions are analogous to the very general assumptions about the continuity and differentiability of various functions in the mechanics of continuous media.

$$(\mathscr{P}_{i\alpha})_+ = (\mathscr{P}_{i\alpha})_-, \qquad (\mathscr{M}_{A\beta})_+ = (\mathscr{M}_{A\beta})_-$$

for $\alpha = 0, 1, \ldots r_1 - 1, \quad \beta = 0, 1 \ldots s_1 - 1$

$$(\mathscr{P}_{i\alpha})_+ = (\mathscr{P}_{i\alpha})_- = 0, \qquad (\mathscr{M}_{A\beta})_+ = (\mathscr{M}_{A\beta})_- = 0 \tag{24}$$

for $\alpha = r_1, \; r_1 + 1, \ldots r - 1, \quad \beta = s_1, \; s_1 + 1, \ldots s - 1$

Conditions (24) can be considered as conditions of continuity (conservation of passage of the world lines of the particles through the discontinuity surface S) of the quantities $\mathscr{P}_{i\alpha}$ and $\mathscr{M}_{A\beta}$ at the discontinuity surface S. This property of the quantities $\mathscr{P}_{i\alpha}$ and $\mathscr{M}_{A\beta}$ constitutes one of their important physical characteristics.

In the more detailed solution of the problems with discontinuous solutions, and especially of problems which involve varying discontinuity surface S (as, for example, with the propagation of isolated dislocations over the particles within the medium, with the growth of cracks, and in other cases) it is possible to generalize basic variational equation (9) and to introduce additional vatiation of the surface S or its edges in the Lagrangian coordinates ξ^i.

Thus, in order to obtain additional relations corresponding to such complicated discontinuity phenomena in real bodies it is generally necessary to complicate the variable functions in basic variational equation (9) by introducing additional terms in δW^* or $\delta\int\Lambda d\tau$ containing the corresponding variations of the Lagrangian coordinates. This is due to the necessity of allowing for the special energy effects associated with the formation or possible propagation of various types of discontinuities over the particles of the medium. These problems will be considered in detail in another paper.

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Translated by A. Y.

STATIC FORMATIONS IN THE GENERAL THEORY OF RELATIVITY AND PLANCKEONS

PMM Vol. 32, №5, 1968, pp. 786-792

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(Received March 27, 1968)

Static homogeneous formations in general relativity theory are considered. It is shown that two types of formations exhaust the possible collection of such formations. The data obtained are to present the planckeon hypothesis of elementary particle structure.

The canonical form of general relativity theory, also known as canonical gravidynamics, variants of which have recently been developed by several authors [1 – 3], permits correct formulation of the general covariant definition of the intrinsic energy of an isolated object, provided the distortion of space-time for which it is responsible, is local. This condition is fulfilled with a high degree of accuracy by elementary particles. It turns out that the intrinsic energy of elementary particles in a gravitational field is finite, and that the domain of definition of an elementary particle must contain a gravitationally self-compensated domain of dimensions $L \sim 10^{-34}$ cm (or, from dimensionality considerations, 10^{-33} cm). The energy included in this domain is on the order of 10^{28} eV (10^{-5} g). The gravitational self-compensation condition implies that the dimensions L of this domain must equal its gravitational radius r_g.